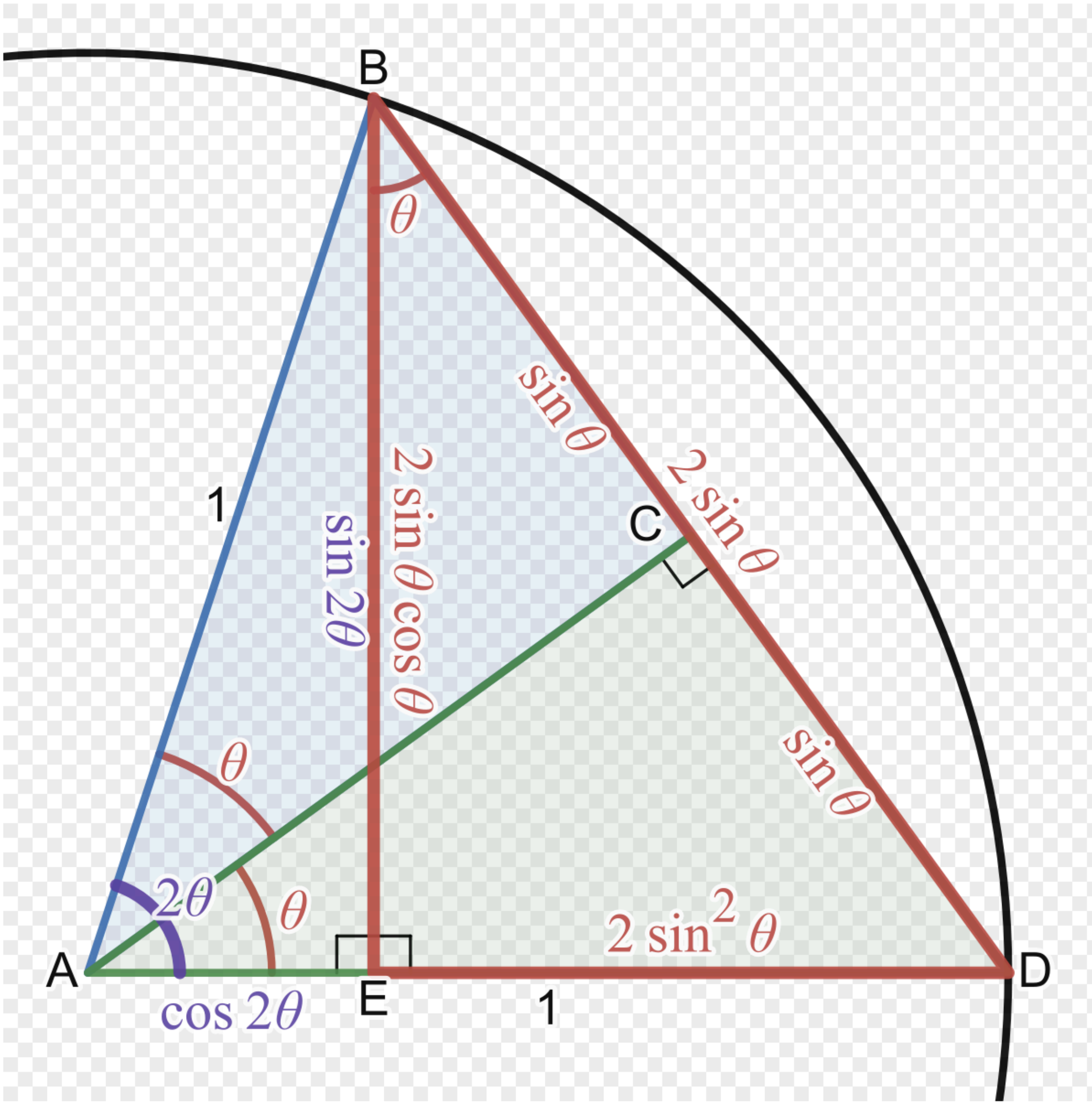
Pre-Calculus



[1]

[1] https://commons.wikimedia.org/wiki/File:Diagram_showing_how_to_derive_the_power_reducing_formula_for_sine.svg

Calculus is the study of continuous change.

We start Calculus with a definition, which you don’t need to know right now. Enjoy the preview.

Definition:

The limit of f (x) , as “x approaches p” , exists and equals L ,

If there exist a δ , such that: $0 < | x - p | < \delta$

implies $0 < | f(x) - L | < \varepsilon$, for all possible ε.

We write:

$$\lim_{x \to p} f(x) = L,$$

x, p, L are all real numbers in the real number system by the way.[2]

This leads to the definition of the **derivative**:

$$L = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

f ‘ (x) equals L.[3]

[2] https://en.wikipedia.org/wiki/Limit_of_a_function#(%CE%B5,_%CE%B4)-definition_of_limit
[3] https://en.wikipedia.org/wiki/Derivative

PreCalculus course work involves the following:

Composite Functions:

$$(g \circ f)(x) = g(f(x)).$$ [4]

For functions f(x) and g(x), "g of f(x)" as we would say, is expressed with the above equation and notation.

Example:

Let $f(x) = 2x$, and let $g(x) = x^2$ to begin with.

$(g \circ f)(x) = g((2x))$.

$= ((2x)^2)$

[4] https://en.wikipedia.org/wiki/Function_composition

Test points:

x	$2x$	$4x^2$
1	2	4
2	4	16
3	6	36

So if we plug in 1 for x into f(x) , our output is 2. Then 2 is passed through g(x) and squared to equal 4.

Polynomial and Rational Functions

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

[5]

n is a non-negative integer that defines the degree of the polynomial. n can be 0, 1, 2, etc.

An example of a polynomial function is

$$y = x^2 + 4x + 4 .$$

The degree of the polynomial is 2.

$a_2 = 1$.
$a_1 = 4$.
$a_0 = 4$.

The a's are constant coefficients.

[5] https://en.wikipedia.org/wiki/Polynomial

If $p(x)$ and $q(x)$ are also polynomial functions,

$$Z(x) = p(x) / q(x) \quad .$$

is also a rational function. $q(x)$ is never equal to 0.

Exponential Functions and Logarithmic Functions

$$b^n = \underbrace{b \times b \times \cdots \times b \times b}_{n \text{ times}}.$$ [6]

b is known as the base and n is called the exponent.

$$y = b^x .$$

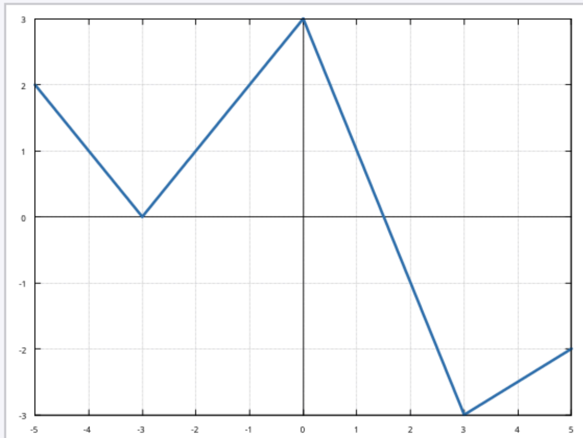
is an exponential function with constant base *b*.

$$\log_b y = x.$$

The logarithmic function above is the inverse function of the general exponential function. The output of a logarithmic function is an exponent.

[6] https://en.wikipedia.org/wiki/Exponential_function

Piecewise Linear Functions



Plot of the piecewise linear function

$$f(x) = \begin{cases} -3 - x & \text{if} \quad x \leq -3 \\ x + 3 & \text{if} \quad -3 \leq x \leq 0 \\ 3 - 2x & \text{if} \quad 0 \leq x \leq 3 \\ 0.5x - 4.5 & \text{if} \quad 3 \leq x \end{cases}$$

[7]

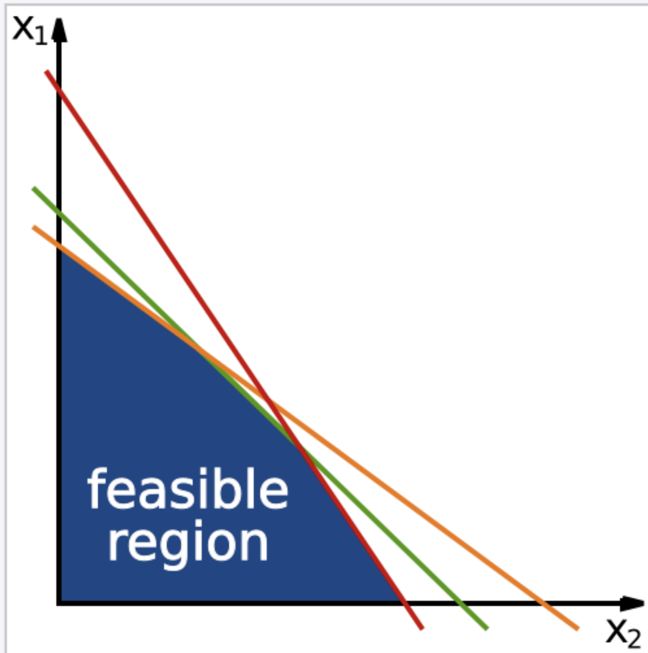
Piecewise functions are what they sound like. For a given piece of an interval, there is an output that doesn’t behave like the other parts of the graph.

For example in the above graph when x is less than or equal to 3, we have a downward sloping line that would intercept the y axis at (0 , -3) . Its slope is -1.

[7] https://en.wikipedia.org/wiki/Piecewise_function

How does the graph behave when our inputs for x are between -3 and 0, including the endpoints?

Inequalties, Linear and Quadratic

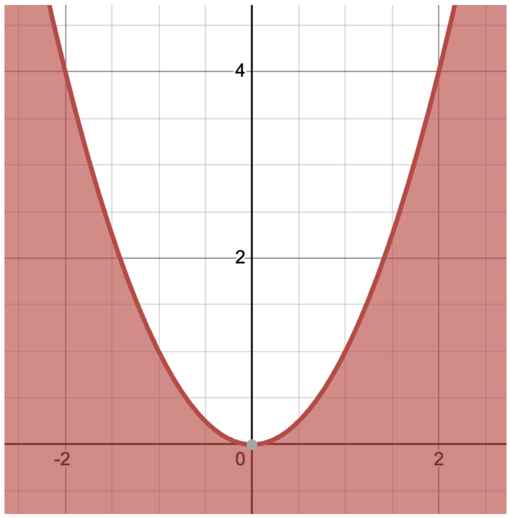


The feasible regions of linear programming are defined by a set of inequalities.[8]

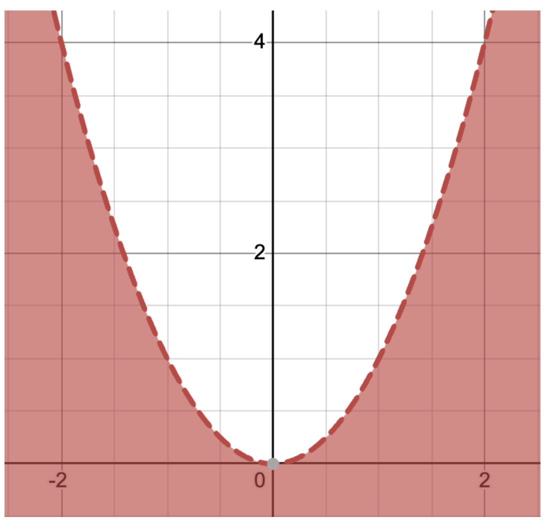
The above graph is just an intuitive picture, we will work with more specific examples.

[8] https://en.wikipedia.org/wiki/Inequality_(mathematics)

<u>Example:</u>

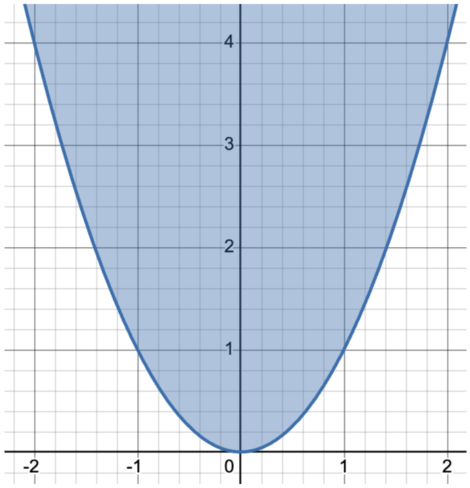


The above graph is of $y \leq x^2$. Note that we shade <u>below</u>, and the line itself is **solid**.

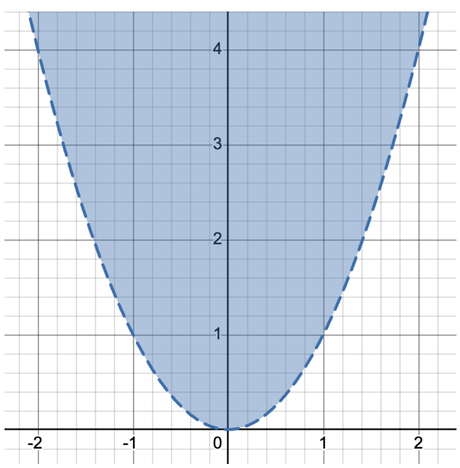


This second graph is $y < x^2$. We again shade <u>below</u>, but the line itself is *dotted*.

Example:

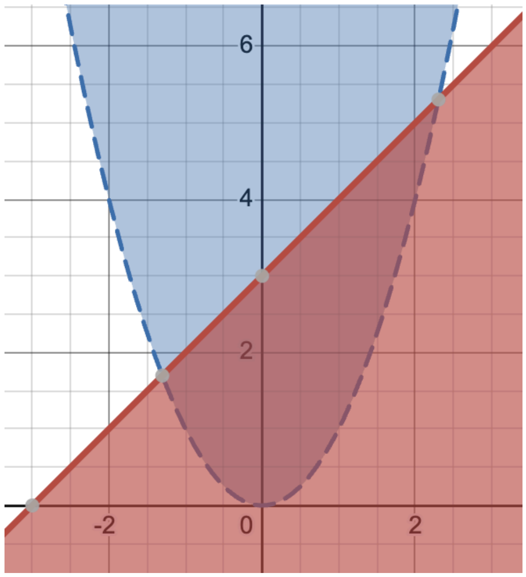


The above graph is of $y \geq x^2$. Note that we shade above, and the line itself is **solid**.

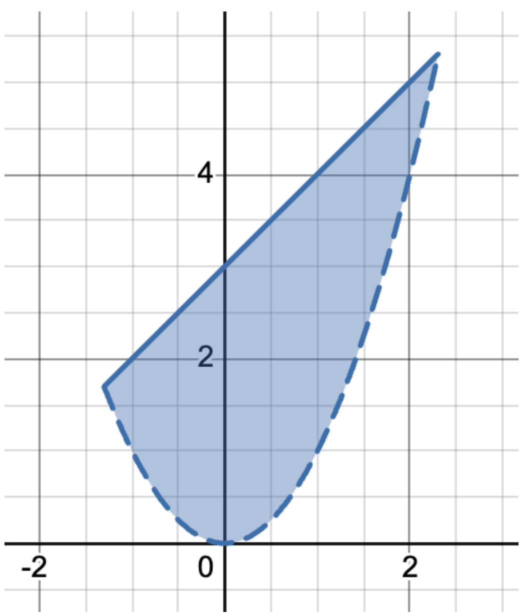


This next graph above is $y > x^2$. Note that we shade above, and the line itself is *dotted*.

Example:



Here we have $y > x^2$ and $y \leq x + 3$.



You would shade like this so to show the solution set of points. Notice how one of the lines is **solid**.

Rational Root Theorem

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$ [9]

Essentially the Rational Root Theorem states that the above rational polynomial of non-negative integer degree n has at most n ***roots***.

Roots are the x-coordinates where the graph crosses the x-axis, the y-coordinate equals 0, and the given polynomial or function equates to 0 for a given x input.

For example:

$x^2 + 4x + 3$ equals 0 ,

when x = -1, or x = -3 .

x = -1 or x = -3 are the roots of the above polynomial.

[9] https://en.wikipedia.org/wiki/Rational_root_theorem

Joint Variation

Example:

$$\varphi\ (x,\, t) = \mathrm{k}xy\ .$$

where x and y are independent variables and k is a constant coefficient.

Trigonometric Functions

sine

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

cosecant

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

secant

$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

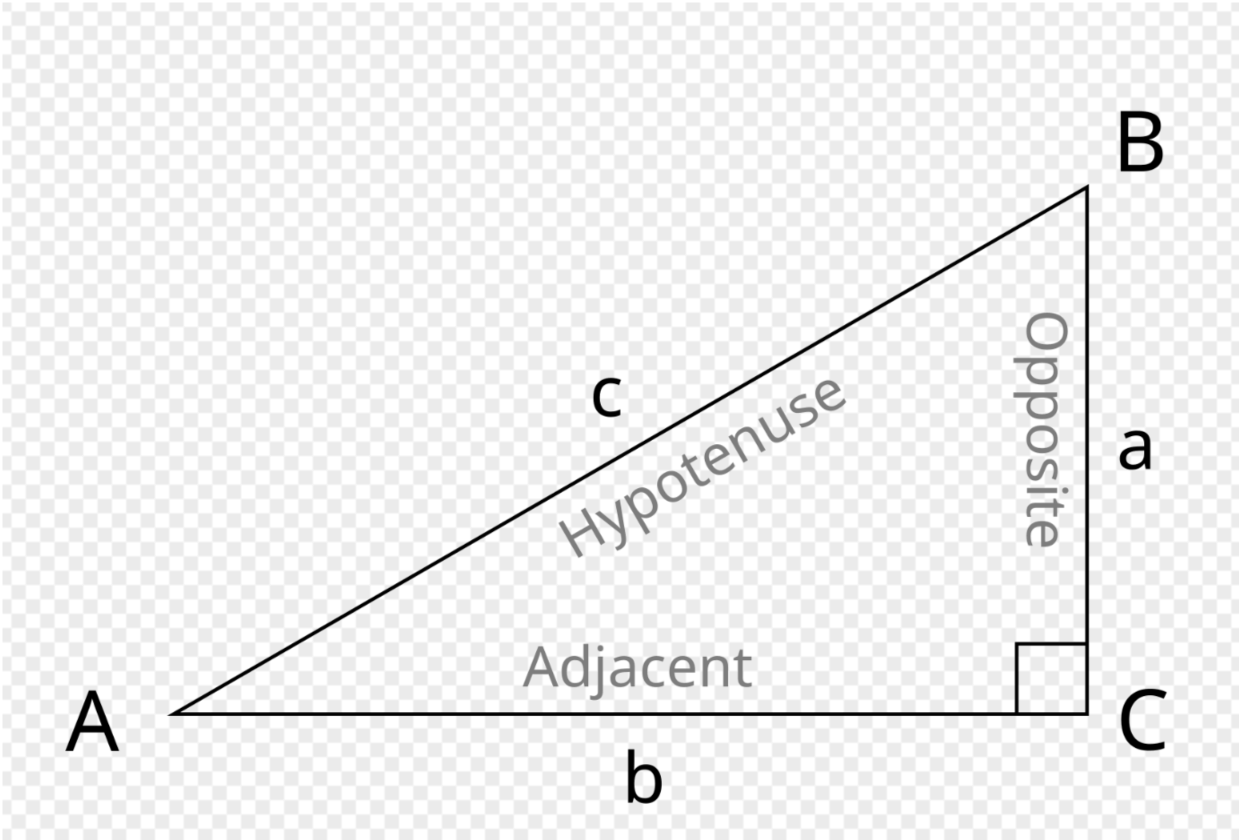
tangent

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

cotangent

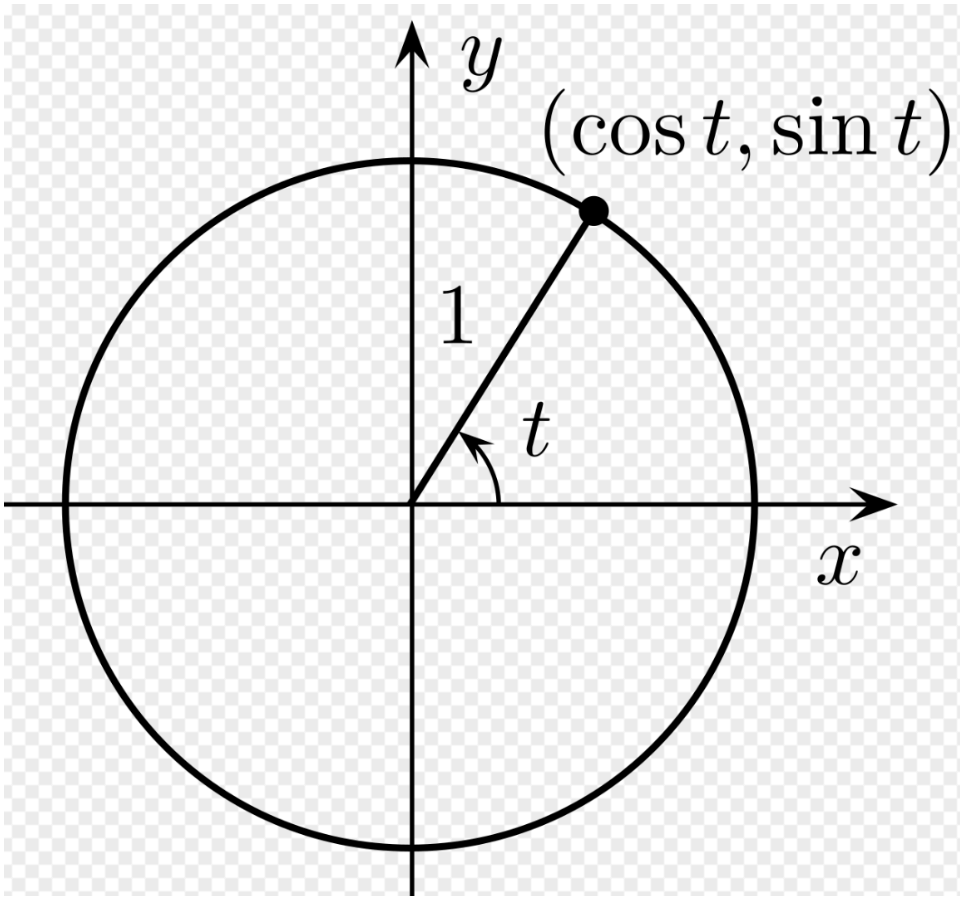
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Adjacent, Opposite, and Hypotenuse refer to the sides around a triangle -as presented on the next page.



Observe angle A at vertex A on the right side of the picture. The sine function of angle a equals the ratio of side a divided by the hypotenuse labeled with the letter c.

sin (A) = opposite / hypotenuse .



[10]

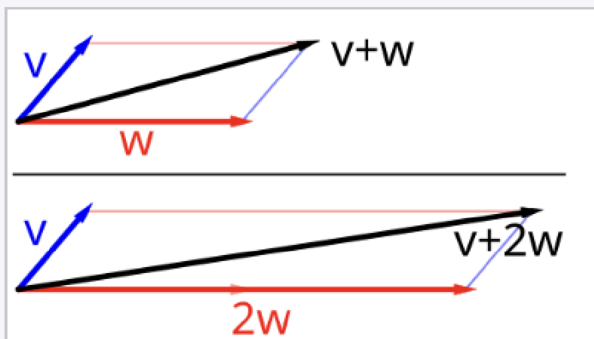
Pythagorean Identity:

$$1 = \sin^2(t) + \cos^2(t) .$$

[10] https://en.wikipedia.org/wiki/Trigonometric_functions

Parameters, Vectors, and Matrices

In math and physics, a **vector** is a term that refers to quantities that cannot be expressed by a single number (a scalar), or to elements of some vector space.[11]



Vector addition and scalar multiplication: a vector v (blue) is added to another vector w (red, upper illustration). Below, w is stretched by a factor of 2, yielding the sum $\mathrm{v} + 2\mathrm{w}$.

Vectors can be added together as indicated by the picture.

[11] https://en.wikipedia.org/wiki/Vector_(mathematics_and_physics)

Matrices

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array}$$

An $m \times n$ matrix: the m rows are horizontal and the n columns are vertical. Each element of a matrix is often denoted by a variable with two subscripts. For example, $a_{2,1}$ represents the element at the second row and first column of the matrix. [12]

A vector is a matrix with one row.

$$\mathbf{a} = (a_1, a_2, a_3, \cdots, a_{n-1}, a_n).$$ [13]

[12] https://en.wikipedia.org/wiki/Matrix_(mathematics)
[13] https://en.wikipedia.org/wiki/Euclidean_vector

The previous vector is of n dimensions in Euclidean space (or $\mathbf{R}^n$) .

Parameters

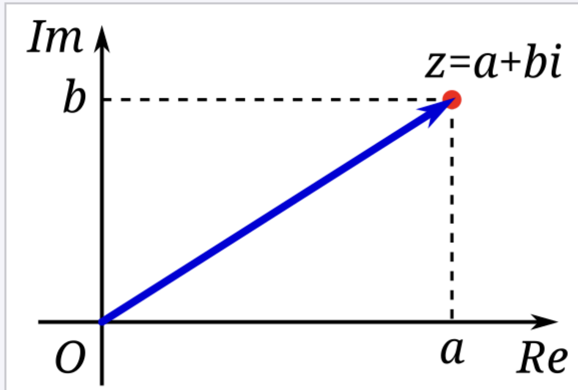
$$f(x) = ax^2 + bx + c;$$

Let's revisit the above quadratic equation. The constant coefficients of a, b, c are referred to as **parameters**.

Parameters are fixed values that help define the nature of the system (in this case a polynomial of degree 2).[14]

[14] https://en.wikipedia.org/wiki/Parameter

Complex Numbers



A complex number z can be visually represented as a pair of numbers (a, b) forming a position vector (blue) or a point (red) on a diagram called an Argand diagram, representing the complex plane. *Re* is the real axis, *Im* is the imaginary axis, and i is the "imaginary unit", that satisfies $i^2 = -1$. [15]

The complex numbers exist on a grid, whereas the real numbers exist on a line.

Consider:

$$1 = -e^{\pi i}, \quad i = \sqrt{(-1)} \quad .$$

[15] https://en.wikipedia.org/wiki/Complex_number

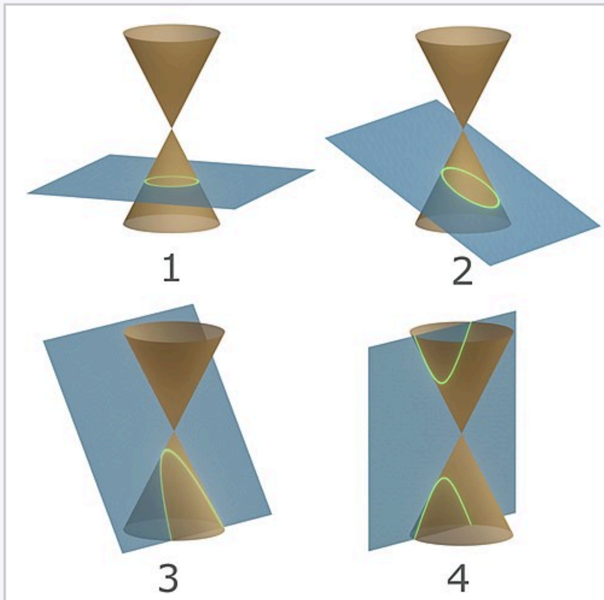
$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots.$$

[16]

Not only is e a letter in the English alphabet, it is also a mathematical constant and number that equals 2.71828… Like π it is transcendental.

[16] https://en.wikipedia.org/wiki/E_(mathematical_constant)

Conic Sections



Types of conic sections:
1: Circle 2: Ellipse
3: Parabola 4: Hyperbola

[17]

[17] https://en.wikipedia.org/wiki/Conic_section

conic section	equation
circle	$x^2 + y^2 = a^2$
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
parabola	$y^2 = 4ax$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

[18]

[18] https://en.wikipedia.org/wiki/Conic_section

conic section	equation	eccentricity (e)	linear eccentricity (c)	semi-latus rectum (ℓ)	focal parameter (p)
circle	$x^2 + y^2 = a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{a^2 - b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 - b^2}}$
parabola	$y^2 = 4ax$	1	N/A	$2a$	$2a$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\sqrt{1 + \frac{b^2}{a^2}}$	$\sqrt{a^2 + b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2 + b^2}}$

[19]

[19] https://en.wikipedia.org/wiki/Conic_section

Probability and Combinatorics

$$P(A) = \frac{\text{(Number of favourable outcomes)}}{\text{(Total number of outcomes)}}$$

For example, heads on a coin toss has a probability of (½) . Tails also has a chance of (½). We'll assume heads or tails are the only two possible outcomes. Together they sum to 1.

Conditional Probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

[20]

[20] https://en.wikipedia.org/wiki/Probability

Example:

Given Event **B** , let's call it: the day starting out hot.

P(**B**) = the probability of a hot day, then.

Next define Event **A** as the act of you buying ice cream on impulse.

P(**A**) = the probability you buy ice cream on impulse.

What is the probability that you'll stop to buy ice cream on impulse (and it is also hot out), given that the day started hot?

P(**A**|**B**) = "probability of: a hot day and you buying ice cream" / "probability the day started out hot."

P (**A** *and* **B**) / P(**B**) .

Series

(Finite and Infinite)

$$s_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n.$$

[21]

Notice that the *n* sums to a finite number, usually one that you just plug in.

$$\sum_{k=0}^{\infty} a_k \quad \text{or} \quad \sum_{k=1}^{\infty} a_k.$$

[22]

These next examples are infinite summations.

[21] https://en.wikipedia.org/wiki/Series_(mathematics)
[22] https://en.wikipedia.org/wiki/Series_(mathematics)

The subtle difference between the n and infinity symbol: ∞ , does make a difference not just in the final answer but sometimes also in the structure of the formula that equates to the given summation.

For now just keep an eye for these different symbols that all look similar.

Limits and Continuity

Definition of Continuity:

$$\lim_{x \to c} f(x) = f(c).$$